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Improper colouring of unit disk graphs

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Abstract: Motivated by a satellite communications problem, we consider a generalised colouring problem on unit disk graphs. A colouring is *k-improper* if no vertex receives the same colour as $k+1$ of its neighbours. The *k-improper chromatic number* $\chi^k(G)$ is the least number of colours needed in a k-improper colouring of a graph G. The main subject of this work is analysing the complexity of computing χ^k for the class of unit disk graph and some related classes, e.g. hexagonal graphs and interval graphs. We show NP-completeness in many restricted cases and also provide both positive and negative approximability results. Due to the challenging nature of this topic, many seemingly simple questions remain: for example, it remains open to determine the complexity of computing χ^k for unit interval graphs.

Key-words: improper colouring, defective colouring, unit disk graph, interval graph, triangular lattice, hexagonal graph, weighted colouring.

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Coloration impropre des graphes d'intersection de disques unitaires

Résumé : Nous modélisons un problème de télécommunications à l'aide des graphes d'intersection de disques unitaires, et d'une généralisation de la notion de coloration. Une coloration est k-*impropre* si chaque sommet a la couleur d'au plus k de ses voisins. Le *nombre chromatique* k-*impropre* $\chi^k(G)$ du graphe G est le plus petit nombre de couleurs pour lequel G admette une coloration k-impropre. Nous étudions la complexité de la détermination de χ^k lorsque l'on se restreint à la classe des graphes d'intersection de disques unitaires, ainsi qu'à d'autres classes de graphes utiles pour la modélisation des problèmes de télécommunications (comme par exemple les graphes hexagonaux). Nous prouvons que ce problème est NP-complet dans de nombreux cas, et fournissons également des résultats positifs et négatifs d'approximation.

Mots-clés : coloration impropre, graphe d'intersection de disques, graphe d'intervalles, réseau triangulaire, graphe hexagonal, coloration pondérée.

1 Introduction

Given a graph G, a colouring c of G, and a subset S of $V(G)$, the *impropriety* of a vertex v restricted to S under c, denoted $\mathrm{im}_S^c(v)$, is the number of neighbours of v in S in the same colour class. We say that a vertex is *proper* in S under c if $\mathrm{im}_S^c(v) = 0$ and *improper* otherwise. The *impropriety* of c in S is $\mathrm{im}_S(c) = \max_{v \in S}\{\mathrm{im}_S(v)\}$. For the above-mentioned notions, we will often drop the superscript or subscript if the context is clear. A colouring is *k-improper* if its impropriety in G is at most k. We say a graph is *k-improperly l-colourable* if it has a k-improper l-colouring. The *k-improper chromatic number* $\chi^k(G)$ is the least l such that G is k-improperly l-colourable.

A unit disk graph is the intersection graph of equal-sized disks in the the plane. In other words, given a set of points fixed in the plane and a positive quantity d, we construct a unit disk graph by adding edges between any two points within distance d of one another. We may assume $d = 1$ or otherwise rescale. In this paper, we study the k-improper colourability problem for unit disk graphs defined for any fixed non-negative integer k as follows.

UD k-IMPROPER CHROMATIC NUMBER
INSTANCE: a unit disk graph G.
QUESTION: what is $\chi^k(G)$?

The main aim is to show that this computational problem is NP-hard. We also consider restricted classes and discuss (in)approximability.

Our motivation for this study is the following problem proposed by Alcatel Space. Satellites send information to stationary receivers on earth. Each receiver is listening on a chosen frequency (or set of frequencies) and can receive information in a certain area (usually modelled by a disk centred at the receiver). The signals of two receivers might interfere if their areas intersect and, in this case, they should normally be listening on different frequencies. However, the intensity of the signal sent by the satellite to a receiver u is supposed to be large near u, and to decrease quickly with distance. Hence, even if the reception areas of receivers u and v intersect, the intensity of the signal sent to v is assumed to be low near u. Furthermore, if u and v have close reception frequencies, then the signal for v contributes interference (otherwise called noise) to the signal received by u, and vice versa. If the total noise does not exceed a certain threshold, then u can still distinguish its signal. We wish to allocate frequencies to the receivers so as to minimise the total bandwidth used.

Like in the case of traditional radio channel assignment, unit disk graphs can be used in a simple model of this problem. We assume that the receivers are scattered in the plane, all receivers have the same reception range and operating bandwidth, and there are no physical obstacles to transmission: the reception areas are uniform disks centred at the receivers. We define an intersection graph as follows: the vertices are the receivers and place an edge between two vertices if their corresponding disks intersect. We furthermore assume that nearby towers contribute noise only if they are on the same channel and that they contribute exactly one unit of noise. We must colour the vertices (i.e. assign a frequency to each receiver) to satisfy the threshold constraint. If we let k be the threshold value (i.e. a receiver can still distinguish its signal if the sum of noises is at most k), then, under this model, our problem is to find an optimal k-improper colouring of the resulting unit disk graph.

1.1 Unit disk graphs

The study of unit disk graphs stems partly from applications in communication networks. The *radio channel assignment problem* (otherwise known as frequency assignment/allocation) is one of the most prominent and well-studied of these applications. For a more general treatment of this topic, consult the lecture series edited by Leese and Hurley [21]. The unit disk colourability problem is one of the simplest models for radio channel assignment and also one of the most well-studied [17, 6, 24, 16, 23].

Classes of graphs related to unit disk graphs

Besides the class of unit disk graphs, we consider some related sub- and superclasses. Denote the class of unit disk graphs by $\mathcal{UD}$. The class of intersection graphs on disks of arbitrary radii in the plane is the class $\mathcal{D}$ of *disk graphs*. This superclass of $\mathcal{UD}$ is relevant to radio channel assignment in that it models radio towers with varying transmission power. Another generalisation of $\mathcal{UD}$ would be to consider higher dimensions; for instance, consider the intersection graph of open (unit) balls in space. We have not yet considered such classes. On the other hand, it is important to consider the restrictions of $\mathcal{UD}$ and $\mathcal{D}$ to one dimension: the class $\mathcal{UI}$ of *unit interval graphs* (a.k.a. *indifference graphs*) and the class $\mathcal{I}$ of *interval graphs* have been extensively studied. These two classes are chordal (hence perfect) classes and many computational problems are feasible when restricted to these classes; consult the monograph by Golumbic [15] for more background into chordal graphs. An important restricted class of unit disk graphs (specific to radio channel) is that of weighted induced subgraphs of the triangular lattice, or *hexagonal graphs* [26, 27, 1]. This class is related to a common placement pattern of radio transmission towers in a cellular communications network: for efficient coverage, the transmitters are only placed on points of a (unit edge length) triangular lattice. We denote the class of hexagonal graphs by $\mathcal{TL}$. We denote the class of planar graphs by $\mathcal{P}$.

Complexity on restriction to unit disk graphs

For channel assignment, three computational problems are of particular interest. Besides CHROMATIC NUMBER — given a graph G, what is the chromatic number $\chi(G)$ of G — the most relevant problems are MAX CLIQUE — given a graph G, what is the clique number $\omega(G)$ of G — and MAX INDEPENDENT SET — given a graph G, what is the independence number $\alpha(G)$ of G. See Table 1 for a summary of what is known about these problems, comparing the restrictions to interval graphs, to unit disk graphs, to planar graphs, to disk graphs and to weighted induced subgraphs of the triangular lattice. Later, we shall be able to add two rows to this table that correspond to k-IMPROPER CHROMATIC NUMBER and MAX k-DEPENDENT SET.

Table 1: Relative complexity for certain problems restricted to the graph classes $\mathcal{I}$, $\mathcal{UD}$, $\mathcal{P}$, and $\mathcal{TL}$.

Problem	$\mathcal{I}$	$\mathcal{UD}$	$\mathcal{P}$	$\mathcal{D}$	$\mathcal{TL}$
CHROMATIC NUMBER	P [28]	NPc [6]	NPc	NPc	NPc [26]
MAX CLIQUE	P [28]	P [6]	P	Open	P
MAX INDEPENDENT SET	P [14]	NPc [6]	NPc	NPc	???

Of particular note is the polynomial-time algorithm for UD MAX CLIQUE. Clark *et al.* [6] exhibited a clever algorithm which utilises the geometry to find subgraphs that can be solved using the polynomial-time algorithm for cobipartite graphs. There is also a polynomial-time algorithm for UD MAX CLIQUE that does not require a representation [31].

Colouring of unit disk graphs

Let us delve into more detail for the problem of computing the chromatic number of unit disk graphs. Clark *et al.* [6] demonstrated NP-completeness of this problem by showing the slightly stronger result that, for the class of unit disk graphs, the computational problem of 3-COL — given a graph G, is G 3-colourable — is NP-complete. The following theorem is a generalisation of this result.

Theorem 1 (Gräf, Stumpf and Weißenfels [16]) *For any fixed integer $l \geq 3$, the problem UD l-COL is NP-complete.*

It is natural to ask how close we can approximate $\chi(G)$. The NP-completeness of UD 3-COL implies that $\chi(G)$ is inapproximable to within a factor of 4/3, unless P = NP. The following proposition implies that $\chi(G)$ is approximable to within a factor of 3.

Proposition 2 (Peeters [30]) *There is a polynomial-time algorithm that, for any unit disk graph G, finds a proper colouring of G using at most $3\omega(G) - 2$ colours. This is a polynomial-time approximation algorithm for UD CHROMATIC NUMBER with performance guarantee of 3.*

The relatively simple proof of this uses geometric ideas to show that the *degeneracy* $\delta^*(G)$ — the supremum of the minimum degree over all induced subgraphs of G — is at most $3\omega(G)-3$; hence, a colouring using at most $3\omega(G) - 2$ colours can be found inductively. (We note that there exist $(3\omega - 3)$-regular unit disk graphs [23].) Similar arguments show that (a) for any unit disk graph G, the maximum degree $\Delta(G)$ is at most $6\omega(G) - 7$, and that (b) for any disk graph G, it holds that $\chi(G) \leq 6\omega(G) - 6$. Proposition 2 was given in 1991; however, there has been no tangible improvement of this approximation result since then. Gräf *et al.* [16] provide a more sophisticated heuristic called the STRIPE algorithm, but it also has performance guarantee of 3. Thus, it is still unknown whether the best approximation ratio for computing the chromatic number of unit disk graphs is closer to 4/3 or 3.

A related problem is to consider the best upper bound on the ratio $\chi(G)/\omega(G)$ for unit disk graphs. Malesińska *et al.* [23] showed that there are classes of unit disk graphs with $\chi(G) \geq \frac{3}{2}\omega(G)$; however, the question of whether this parameter is closer to 3/2 or 3 is an enticing open problem. Results on colouring of random unit disk graphs show that this parameter is lower than 3 for "most" unit disk graphs [25].

Colouring of hexagonal graphs

In light of the particular application of frequency assignment in cellular telephone networks, McDiarmid and Reed [26] studied the weighted colouring problem on induced subgraphs of the triangular lattice T.

Given a graph G, a *weight assignment* w is an association of each vertex $v \in V(G)$ with a non-negative weight w_v. A *weighted graph* is an ordered pair (G, w) where G is a graph and w is a weight assignment for G. A *weighted colouring* c of a weighted graph (G, w) is an assignment to each vertex v of a multiset $c_w(v)$ of w_v colours. A *weighted l-colouring* is a weighted colouring where the colours are chosen from $\{1, \ldots, l\}$. (We deviate slightly from the definition in McDiarmid and Reed [26] to eventually allow impropriety.) A weighted l-colouring of (G, w) is *proper* if each multiset $c_w(v)$ is a set of w_v distinct colours, and adjacent vertices receive disjoint multisets c_w.

Note that, given a weighted unit disk graph (G, w), there is a natural corresponding graph G_w obtained by replacing each vertex v of G by a clique of size w_v. If G is a unit disk graph, then G_w is also a unit disk graph. Also, the above definitions associated with weighted (proper) colouring of (G, w) correspond directly with those associated with (proper) colouring of G_w.

McDiarmid and Reed [26] proved that the weighted (proper) 3-colourability problem restricted to hexagonal graphs is NP-complete. They also provided a 4/3-approximation algorithm for the weighted chromatic number of such graphs.

1.2 Improper colouring

The main aim of this paper is to establish results analogous to those just mentioned, but for improper colouring. This natural generalisation of colouring was introduced by Cowen, Cowen and Woodall [8] (however, they used the terms *(l, k)-colourable* and *defective*). Note that 0-improper colouring is exactly proper colouring; hence, the 0-improper chromatic number $\chi^0(G)$ is precisely the chromatic number $\chi(G)$. Also observe that, if a graph is k-improperly l-colourable, then it is k_1-improperly l_1-colourable for all integers k_1 and l_1 such that $k_1 \geq k$ and $l_1 \geq l$.

An alternative definition of improper colouring is as follows. A subset of $V(G)$ is called *k-dependent* if it induces a subgraph of G with maximum degree at most k. A graph is k-improperly l-colourable if its vertex set can be partitioned into l parts each of which are k-dependent. It is natural to consider the related *k-dependence number* $\alpha^k(G)$, the size of the largest k-dependent set in G. The maximum k-dependent set problem was first considered by Garrido *et al.* [10]. Note that a 0-dependent set is simply an independent set; hence, the 0-dependence number $\alpha^0(G)$ is exactly the independence number $\alpha(G)$.

Basic results on improper colouring

We present here some basic general bounds for the k-improper chromatic number. Recall two elementary lower bounds for χ in terms of the clique and independence numbers.

Proposition 3 *For any graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$.*

The following is an immediate generalisation of this last proposition to k-improper colouring.

Proposition 4 *For any graph G and any non-negative integer k, $\chi^k(G) \geq \frac{\chi(G)}{k+1}$ and $\chi^k(G) \geq \frac{|V(G)|}{\alpha^k(G)}$.*

Similarly, the following is a familiar upper bound in terms of the maximum degree.

Proposition 5 *For any graph G, $\chi(G) \leq \Delta(G) + 1$.*

Consider the following approximation result for graphs of bounded maximum degree.

Proposition 6 (Lovász [22]) *Let G be a graph with maximum degree $\Delta \leq l(k+1) - 1$ for fixed non-negative integers k and l. There is an algorithm to k-improperly l-colour G in $O(\Delta|E(G)|)$ time.*

Now the k-improper analogue to Proposition 5 is a corollary.

Corollary 7 *For any graph G and any non-negative integer k, $\chi^k(G) \leq \left\lceil \frac{\Delta(G)+1}{k+1} \right\rceil$.*

We implicitly observed near the end of the last subsection that $\chi(G) \leq \delta^*(G) + 1$ for every graph G; however, no improper analogue of this result holds.

Improper colouring of planar graphs

Improper colouring for planar graphs was first considered by Cowen, Cowen and Woodall [8], where they attempted to generalise the Four Colour Theorem [2, 4]. They asked: for fixed k, what is the smallest p_k such that every planar graph is k-improperly p_k-colourable? It was pleasantly surprising that this question could be fully answered, in particular, that $p_0 = p_1 = 4$ and $p_k = 3$ for all $k \geq 2$.

Here, we should mention a related problem that is of continuing interest. The list colouring analogue of the above problem is informally stated as follows: what is the smallest p_k^* such that every planar graph is k-improperly p_k^*-choosable? Due to Voigt [35] and Thomassen [34], we know that $p_0^* = 5$; independently, Eaton and Hull [11] and Škrekovski [33] showed that $p_2^* = 3$; however, it is is still open whether p_1^* is 4 or 5.

It was not until a decade after the concept first appeared when Cowen, Goddard and Jesurum [7] considered the complexity of improper colouring of (planar) graphs. The proof of the Four Colour Theorem naturally extends to an algorithm for 4-colouring a planar graph [3]; indeed, a quadratic algorithm has been developed [32]. The proof of planar 2-improper 3-colourability [8] naturally extends to a linear algorithm. It is well-known that it is NP-complete to determine if a planar graph is 3-colourable and that 2-colouring is polynomial-time for general graphs. The remaining complexity questions are answered as follows.

Theorem 8 (Cowen *et al.* [7])

1. *It is NP-complete to determine if a planar graph is 1-improperly 3-colourable.*
2. *For any fixed positive integer k, it is NP-complete to determine if a planar graph is k-improperly 2-colourable.*

Furthermore, for general graphs, it is known that (like for proper colouring) there exists $\varepsilon > 0$ such that χ^k cannot be approximated to within a factor of n^ε, unless P = NP.

2 Main results

2.1 Improper colouring of unit disk graphs

In this subsection, we present our analysis for the unit disk improper colourability problem.

Since the l-colourability problem for unit disk graphs is NP-complete for any fixed integer $l \geq 3$ (cf. Theorem 1), we expect that the corresponding k-improper l-colourability problem is also NP-complete for any fixed positive integer k. Our expectation is correct and we use a reduction similar to that of Gräf *et al.* [16], i.e. from l-colourability.

Theorem 9 *Unit disk k-improper l-colourability is NP-complete for any fixed integers k and l such that $k \geq 0$ and $l \geq 3$.*

Our approach generalises that of Gräf *et al.* [16] and our key contribution is to produce auxiliary graphs — in particular, the crossing auxiliary graph — that are more general than those of the proof for unit disk l-colourability.

It is not clear if we should expect k-improper 2-colourability for unit disk graphs to be NP-complete, as 2-colourability is polynomial-time in general, while the planar k-improper 2-colourability problem, for any fixed positive integer k, is NP-complete (cf. Theorem 8). We can in fact reduce from the latter problem to show NP-completeness.

Theorem 10 *Unit disk k-improper 2-colourability is NP-complete for any fixed positive integer k.*

The reduction from planar k-improper 2-colourability requires no crossing auxiliary graphs. However, the auxiliaries must transmit information about impropriety; also, we need to take care of high-degree vertices. The task of constructing such auxiliary graphs is the crux of the reduction.

A detailed proof of Theorems 10 and 9 can be found in Appendix A. These two results show that, like for unit disk (proper) colourability, the unit disk improper colourability problem is NP-hard in a relatively strong sense. In light of these negative results, our next question is to consider approximability. By Theorem 10, the k-improper chromatic number (for $k \geq 1$) is inapproximable to within 3/2, unless P = NP. Since there is no improper analogue for colouring graphs with bounded degeneracy, the only known positive approximation result is the following.

Proposition 11 *For any fixed non-negative integer k, there is a polynomial-time approximation algorithm that, given a unit disk graph G, finds a k-improper colouring of G using at most $\left\lceil \frac{6\omega(G)-6}{k+1} \right\rceil$ colours. This is a polynomial-time approximation algorithm for UD k-IMPROPER CHROMATIC NUMBER with performance guarantee of 6.*

This is a direct consequence of the bound $\Delta(G) \leq 6\omega(G) - 7$ for unit disk graphs, and Propositions 4 and 6. It is unknown whether the best approximation ratio for computing χ^k of unit disk graphs is closer to 3/2 or 6, if k is a fixed positive integer.

A related problem is to consider the best upper bound for unit disk graphs on the ratio between the k-improper chromatic number and the trivial lower bound, i.e. $\left\lceil \frac{\omega}{k+1} \right\rceil$. We mentioned at the end of Subsection 1.1 that for $k = 0$ this bound is between 3/2 and 3. By the last proposition, we know that for positive integers k this bound is at most 6.

Proposition 12 *There exist unit disk graphs G_n such that*

$$\frac{\chi^k(G_n)}{\omega(G_n)/(k+1)} \geq \begin{cases} 2 & \text{if } k \text{ is odd, and} \\ 2(k+1)/(k+2) & \text{if } k \geq 2 \text{ is even.} \end{cases}$$

Proof. Fix an arbitrary integer $n > k/2 + 1$. Consider the graph G_n whose vertices are the $2n$ points equally spaced on a circle. Join each point to all other points on the circle except for the one directly opposite it. It can be verified that G_n is a unit disk graph and that $\omega(G_n) = n$. Since each vertex is adjacent to all but one vertex in G_n, there can be no k-dependent set with more than $k+2$ vertices, giving that $\alpha^k(G_n) \leq k+2$. When k is odd, we can reduce this estimate by one: suppose k is odd and there is a k-dependent set S of size $k+2$. Since k is odd, there must be two opposite (non-adjacent) vertices u and v such that only one of them, say v, is in S. Then v must be adjacent to all other vertices in S and hence have degree $k+1$, a contradiction. We now apply Proposition 4 and the result follows. □

These examples are inspired by the unit disk graphs that show the ratio 3/2 can be attained in the case $k=0$ (cf. Malesińska *et al.* [23]) — these are also formed by equally spaced points around a circle. That we can obtain higher ratios for all other cases (except $k=3$) gives us further evidence to believe that, for unit disk graphs, the improper chromatic numbers (i.e. when $k \geq 1$) are harder to approximate that the chromatic number.

On the other hand, we note that random analysis for this problem, generalising the results of McDiarmid [25], has been performed [20]. In the standard model for random unit disk graphs, it is shown, for nearly all asymptotic choices for the distance parameter $r(n)$, that, as $n \to \infty$, the k-improper chromatic number tends to a value at most $2\sqrt{3}/\pi \approx 1.103$ times the optimal. One interpretation of this is that, given large instances G of randomly generated unit disk graphs, returning $2\sqrt{3}/\pi \cdot \frac{\omega(G)}{k+1}$ is a reasonable approximation for χ^k.

2.2 Improper colouring of hexagonal graphs

In this subsection, we investigate two variants of weighted improper colouring on induced subgraphs of the triangular lattice T.

A weighted colouring c of (F,w) is *k-improper* if, for every vertex $v \in V(F)$ and each colour $x \in c_w(v)$, the number of times (counted with multiplicities) the colour x is assigned to v and to any neighbour of v is at most $k+1$. We define the *weighted k-improper chromatic number* $\chi(F,w)$ to be the least l needed in a weighted k-improper l-colouring. Note that $\chi^k(F,w)$ is precisely $\chi^k(F_w)$, the improper chromatic number of the corresponding unweighted graph. We consider the weighted k-improper colouring problem on the triangular lattice and, for any fixed non-negative integers k and l, we define the following computational problem.

TL k-IMPROPER l-COL
INSTANCE: an induced subgraph F of the triangular lattice T together with a corresponding weight assignment w.
QUESTION: does (F,w) have a weighted k-improper l-colouring?

We show NP-completeness of this problem for $l=3$.

Theorem 13 *For any fixed non-negative integer k, TL k-IMPROPER 3-COL is NP-complete*

The proof, given in Appendix A, is a generalisation of the NP-completeness proof of McDiarmid and Reed [26], and is a reduction from 3-colourability of planar graphs with maximum degree 4. This is the shortest proof of the fact that computing the k-improper chromatic number of unit disk graphs is NP-hard.

Let us now consider approximability for computation of χ^k for hexagonal graphs. The previous theorem shows that χ^k is inapproximable to within 4/3 for such graphs, unless P = NP. We point out that a 3-approximation algorithm follows from Propositions 4 and 6 and the fact that $\Delta(F_w) \leq 3\omega(F_w) - 3$ for any $\mathcal{TL}$ graph (F,w); however, we can obtain a better approximation algorithm by pursuing the method in the second half of the paper by McDiarmid and Reed [26].

Theorem 14 *For any fixed non-negative integer k, there is a polynomial-time approximation algorithm that, given a weighted induced subgraph (F,w) of the triangular lattice, finds a weighted k-improper colouring of F using at most $\left(\frac{4\omega(F_w)+1}{3} + 4(2k+1)\right)/(k+1)$ colours.*

When $\omega(F)$ is large relative to k, this approximation algorithm is essentially optimal for χ^k of $\mathcal{TL}$ graphs. This result suggests that TL k-IMPROPER 2-COL should be polynomial, but we have been unable to exhibit an algorithm to justify this claim.

Proof Outline. We will closely follow the approach of McDiarmid and Reed [26] and the reader is advised to consult that paper for the details that have been omitted. We begin by giving an analogue of Lemma 1 in McDiarmid and Reed [26].

Lemma 15 *For any fixed non-negative integer k, there is a polynomial-time algorithm that, given a weighted bipartite graph (G, w), finds a weighted k-improper colouring of G using at most $\hat{l} = \left\lceil \frac{\omega(G_w)}{k+1} \right\rceil + 1$ colours.*

Proof. Let $G = (A, B)$ be the bipartition and determine $\omega(G_w)$. To specify a *near* optimal weighted colouring, let $T_v = \left\{1, \ldots, \left\lceil \frac{w_v}{k+1} \right\rceil \right\}$ if $v \in A$ and $T_v = \left\{\hat{l} - \left\lceil \frac{w_v}{k+1} \right\rceil + 1, \ldots, \hat{l}\right\}$ if $v \in B$. If $u \in A$ and $v \in B$ are adjacent, then $\left\lceil \frac{w_v}{k+1} \right\rceil + \left\lceil \frac{w_v}{k+1} \right\rceil \leq \hat{l}$ so that $T_u \cap T_v = \emptyset$. Thus, for each vertex v, it is adequate to assign colours from T_v with multiplicity up to $k+1$ to obtain a weighted k-improper colouring. $\Diamond$

For the rest of the proof, we follow the proof of Theorem 2 in McDiarmid and Reed [26], except for a few modifications. Their proof is a two-stage colouring procedure, using 3κ colours in the first stage and, in the second stage, applying their Lemma 1 to optimally colour the remaining weighted bipartite graph U, using at most $\omega(U) \leq \omega(F_w) - 2\kappa$ additional colours, where $\kappa = \left\lfloor \frac{\omega(F_w)+1}{3} \right\rfloor$. The modifications we will make are to use $3 \cdot \left(\left\lceil \frac{\kappa}{k+1} \right\rceil + 1 \right)$ colours (using a similar method to the above lemma) in the first stage and, in the second stage, apply our lemma above to U, using at most $\left\lceil \frac{\omega(F_w)-2\kappa}{k+1} \right\rceil + 1$ additional colours. Some routine integer-part calculations show that the number of colours used overall is at most the announced quantity. $\Box$

The second type of weighted colourings we consider are those in which the colour multisets $c_w(v)$ consist of w_v distinct colours. We call these *weighted distinct k-improper colourings.* We see that (F, w) is distinct $\Delta(F)$-improperly l-colourable if $l \geq \max\{w_v : v \in F\}$. We consider the weighted distinct k-improper colouring problem on the triangular lattice and, for any fixed non-negative integers k and l, we define the following computational problem.

TL DISTINCT k-IMPROPER l-COL
INSTANCE: an induced subgraph F of the triangular lattice T together with a corresponding weight assignment w.
QUESTION: does (F, w) have a weighted distinct k-improper l-colouring?

The practical consideration for this problem is that if each point of the lattice is thought of as being a radio transmission tower, then there will doubtless be conflict if it tries to broadcast two messages on the same frequency. We again only consider the case $l = 3$. If $k = 0$, the problem is clearly NP-complete by Theorem 13, and the reader can also refer to the paper of McDiarmid and Reed [26]. If $k \geq 6$, then, since the triangular lattice is 6-regular, the problem is trivial. We show that the problem stays intractable until $k = 6$.

Theorem 16 *For fixed $k \in \{0, \ldots, 5\}$, TL DISTINCT k-IMPROPER 3-COL is NP-complete.*

The proof of this theorem is found in Appendix A.

2.3 Improper colouring of interval graphs

In this subsection, we investigate improper colouring for interval and unit interval graphs.

Proposition 17 *For any fixed non-negative integers k and l, there exists a unit interval graph $I_{k,l}$ with maximum degree and clique number equal to $l(k+1)$ which is not k-improperly l-colourable.*

Proof. To construct $I_{k,l}$, just start with a $(l(k+1))$-clique $K = K_{l(k+1)}$ and add a vertex u linked to exactly $(l-1)(k+1)+1$ vertices of K. The obtained graph is a unit interval graph.

Suppose that $I_{k,l}$ has a k-improper l-colouring: K must have exactly $(k+1)$ vertices of each colour. Thus any vertex of K has impropriety k in K. As u has $(l-1)(k+1)+1$ neighbours in K it must have at least one neighbour of each colour and hence cannot be coloured, a contradiction. $I_{k,l}$ is clearly a unit interval graph. □

This proposition raises the question of the complexity of k-improperly l-colouring unit interval graphs for fixed non-negative integers k and l. We prove now that this problem is polynomial time for general interval graphs, and we provide a dynamic programming algorithm.

Theorem 18 *The k-improper l-colourability problem restricted to interval graphs is in P for any fixed non-negative integers k and l.*

Proof. Let G be an interval graph. We preprocess the graph by computing $\omega(G)$ (and this can be done in polynomial time). We may assume that $\omega(G) \leq l(k+1)$; otherwise, G is not k-improperly l-colourable by Proposition 4. Now assume we have an interval representation for G. Let $v_1, \ldots, v_n$ be the vertices of G ordered by the left endpoints of the respective intervals. We consider the vertices one-by-one according to this order and assign v_1 colour 1.

For this algorithm, we maintain all valid partial k-improper l-colourings of the induced subgraph processed so far; however, we discard vertices that are not required. More precisely, suppose v records the next vertex to be processed and we wish to extend all of the partial colourings (and discard ones that are impossible to extend). We need only maintain a list of all valid partial k-improper l-colourings (together with accumulated improprieties) of a set S, where S contains all previously coloured neighbours of v.

If the vertex v_j is not adjacent to $v = v_s$, where $j < s$, then v_j is not adjacent to v_i with $i \geq s$ (and hence we can safely remove v_j from S). Furthermore, the maximum number of vertices in S at any given point in time is $\omega - 1$, since S together with v induces a clique. Thus, a list of size $(lk)^{\omega(G)} \leq (lk)^{l(k+1)}$ is sufficient. Note that the step of colouring a vertex and updating the list is polynomial in time. □

This result does not fully answer the complexity question for improper colouring of unit interval graphs: it is unknown whether, for $k > 0$ fixed, there is a polynomial-time algorithm to find $\chi^k(G)$ given a unit interval graph G. The following result, however, shows that only two values are possible: the lower bound given by Proposition 4, or this number plus one.

Theorem 19 *For any fixed non-negative integer k, there is a linear-time algorithm that, given a unit interval graph G, finds a k-improper colouring of G using at most $\hat{l} = \left\lceil \frac{\omega(G)}{k+1} \right\rceil + 1$ colours.*

Proof. Let $v_1, \ldots, v_n$ be a unit interval representation for G. Under this ordering, our colouring procedure proceeds by assigning colour 1 to the first $k+1$ vertices, colour 2 to the next $k+1$, and so on until colour $\hat{l}$ has been assigned whereupon it begins assigning colour 1 again. If we now have an invalid colouring, we can suppose without loss of generality that v_{k+1} and $v_{(k+1)\hat{l}+1}$ (both coloured 1) are adjacent. But, because G is a unit interval graph, this implies that $\left\{v_{k+1}, \ldots, v_{(k+1)\hat{l}+1}\right\}$ induces a clique in G and this contradicts the choice of $\omega(G)$. □

When only k is fixed, one can think of applying the algorithm of Theorem 18 with $l = \left\lceil \frac{\omega}{k+1} \right\rceil$. However, this may be polynomial neither in space nor in time, since space and time complexity both are $\mathcal{O}(lk)^{l(k+1)}$ and ω can be linear in the number of vertices. In light of this, we pose the following problem.

Problem 20 *Let G be a unit interval graph, and k a positive integer. The preceding result states that $\chi^k(G) \in \left\{\left\lceil\frac{\omega(G)}{k+1}\right\rceil, \left\lceil\frac{\omega(G)}{k+1}\right\rceil + 1\right\}$. Is there a polynomial-time algorithm to decide which value is correct?*

The following problems also remain open.

Problem 21 *For any fixed positive integer k, is there a polynomial-time algorithm that, given an interval graph G, computes $\chi^k(G)$?*

Problem 22 *For any fixed positive integer k, what is the largest ratio between χ^k and $\omega/(k+1)$ for interval graphs?*

2.4 Maximum k-dependent set for unit disk graphs

Since $\chi^k(G) \geq |V(G)|/\alpha^k(G)$, computing the k-dependence number could be helpful in finding a lower bound for the k-improper chromatic number. Dessmark, Janson and Lingas [9] were the first to study the problem of computing the size of a maximum k-dependent set. Among other results, they showed that MAX k-DEPENDENT set for bipartite planar graphs is NP-complete. It is natural to expect the same for unit disk graphs.

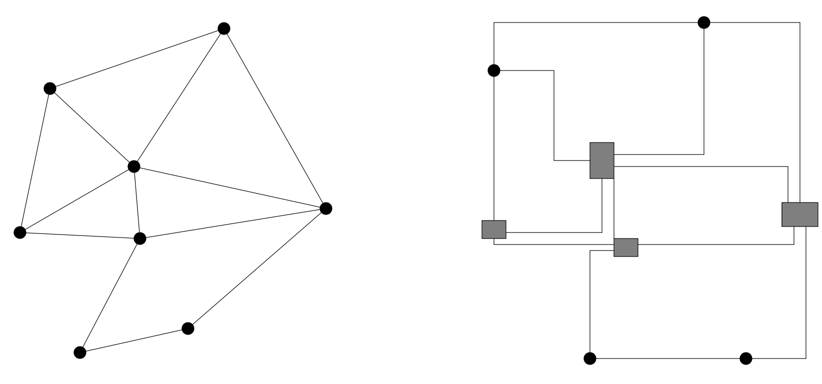
Theorem 23 *MAX k-DEPENDENT SET for unit disk graphs is NP-complete, for any fixed non-negative integer k.*

Note. The following reduction uses a special embedding which is used in the proof of Theorem 10.

Proof. Our reduction is from PLANAR INDEPENDENT SET. Let G be a planar graph. Our first step is to find a special embedding of G that is realisable as a unit disk graph.

The reductions of Clark *et al.* [6] use an orthogonal embedding of G, i.e. a planar embedding of G such that each edge corresponds to an arc made up of horizontal and vertical line segments, but this requires that G has maximum degree at most 4. In Gräf *et al.* [16], each edge corresponds to an arc made up of horizontal and vertical line segments in the embedding of G; however, G need not be planar and, to take account of high-degree vertices, each vertex is represented by a (possibly degenerate) line segment. Here, we use what is called a box-orthogonal embedding.

A *box-orthogonal embedding* of G is a planar embedding of G such that each edge is represented by alternate horizontal and vertical line segments and each vertex is represented by a (possibly degenerate) rectangle, called a box (See Figure 1). We assume that all line segments, including those at the perimeter of a box, lie on lines of the integer grid. There is a box-orthogonal embedding for each planar graph and one can be computed in polynomial time [12, 29].



(a) (b)

Figure 1: (a) An arbitrary planar graph G and (b) a box-orthogonal embedding of G.

Now, given G, generate a box-orthogonal embedding. To make all line segments of even length, we subdivide the axes. For each vertex $v \in V(G)$, we have replaced v by a box $Box(v)$. The perimeter of $Box(v)$ is incident with $\deg(v)$ line segments. Pick two points of incidence that are adjacent on the perimeter and remove the segment(s) joining them. At each of the $2b(v)+1$ grid points that lie on the remaining portions of the perimeter of $Box(v)$, place exactly $k+1$ unit disks. For each edge $e \in E(G)$, we have replaced e by alternating horizontal and vertical line segments. At each of the $2l(v)$ grid points contained in one of these line segments, again place $k+1$ unit disks.

This construction takes polynomial time and one can verify that the resulting unit disk graph $\hat{G}$ has a k-dependent set of size $m + \sum_{v \in V(G)} b(v) + \sum_{e \in E(G)} l(v)$ if and only if G has an independent set of size m. □

Note that the reduction just used is actually to the MAX k-DEPENDENT SET for weighted induced subgraphs of the grid graph. It is technically possible (but labourious) to show a similar reduction to MAX k-DEPENDENT SET for the class $\mathcal{TL}$.

Despite this hardness result, we can make a couple of positive remarks. First, there is a polynomial-time approximation scheme (PTAS) for UD MAX k-DEPENDENT SET.

Proposition 24 *Given $\varepsilon > 0$, there is a polynomial-time algorithm which can approximate UD MAX k-DEPENDENT set to within a factor of $1-\varepsilon$.*

Proof. This is done by splitting the plane into large disconnected square pieces and solving exactly on those pieces (cf. Hunt *et al.* [19]). □

Second, there is a polynomial-time algorithm to compute the 1-dependence number for a class which strictly contains interval graphs. A graph G is *weakly chordal* if neither G nor its complement contains an induced cycle of length at least five. This (perfect) class was introduced by Hayward [18] and contains chordal and hence interval graphs.

Proposition 25 *There is a polynomial-time algorithm that, given a weakly chordal graph G, finds a maximum 1-dependent set.*

Proof Outline. This mimics the proof by Cameron *et al.* [5] that there is a polynomial-time algorithm to find maximum induced matchings. □

It remains to determine the complexity of MAX k-DEPENDENT SET for interval graphs (let alone weakly chordal graphs) for fixed integers $k \geq 2$.

3 Concluding remarks

We have established NP-completeness in nearly all cases of the decision version of the improper colourability problem of unit disk graphs. For weighted colouring of hexagonal graphs, we showed NP-completeness for all possible values of k, even when we considered the distinct variant, but we have not analysed the decision problem for $l > 3$.

Table 2: An update to Table 1

Problem	$\mathcal{I}$	$\mathcal{UD}$	$\mathcal{P}$	$\mathcal{D}$	$\mathcal{TL}$
CHROMATIC NUMBER	P	NPc	NPc	NPc	NPc
MAX CLIQUE	P	P	P	Open	P
MAX INDEPENDENT SET	P	NPc	NPc	NPc	NPc
MAX k-DEPENDENT SET	Open	NPc	NPc	NPc	NPc
k-IM CHROMATIC NUMBER	Open	NPc	NPc	NPc	NPc

On the positive side, there is a 6-approximation for the k-improper chromatic number of unit disk graphs and an approximation within 1 for unit interval graphs. There is a straightforward 3-approximation and a more-complicated, "near" 4/3-approximation for computing the weighted k-improper chromatic number of hexagonal graphs.

However, these positive results are unsatisfying to us. We believe that the approximation ratio for unit disk graphs should be closer to 3, if not lower. Also, the complexity of determining the k-improper chromatic number for unit interval graphs is still unknown.

We showed that MAX k-DEPENDENT SET is NP-complete for unit disk graphs, and even for weighted induced subgraphs of the square or triangular lattices. Also, this problem allows a PTAS. For interval graphs, MAX 1-DEPENDENT SET is polynomial-time solvable, but the complexity is open for higher values of k.

We end with two tempting problems.

Problem 26 *Is the following problem NP-complete?*
INSTANCE: *an induced subgraph G of the triangular lattice.*
QUESTION: *is the graph G 1-improperly 2-colourable?*

Problem 27 *For any fixed positive integer k, is there a polynomial-time approximation algorithm for computing the k-improper chromatic number of disk graphs with constant approximation ratio?*

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A NP-completeness proofs

To aid understanding of the proof outlines in Subections A.1 and A.2, we give brief overviews of the proofs of NP-completeness for UD 3-COL in Clark *et al.* [6] and of Theorem 1.

The proof for NP-completeness of UD 3-COL given in Clark *et al.* [6] is a reduction from PLANAR 3-COL for graphs with maximum degree at most 4. The most important section of this proof is the polynomial-time generation of a restricted planar embedding such that the edges are only drawn on lines of the integer grid (hence the need for graphs with maximum degree at most 4). Subsequently, the embedded edges are replaced by sequences of unit disk graphs that "communicate" the 3-colourability of the original graph. This is a relatively standard NP-completeness reduction technique known as local replacement [13].

For Theorem 1, Gräf *et al.* [16] attempt to generalise this approach; however, the first main difficulty is that reduction from PLANAR l-COL for $l > 3$ is impossible due to the four-colour theorem. Instead, the reduction is from general l-COL, but this introduces two difficulties: higher degrees and crossing edges. To handle the first difficulty, they use a special (non-planar) graph embedding in which the edges are still drawn on lines of the integer grid, but each high-degree vertex v is replaced by a set of points $M(v)$ of size $\deg(v)$ spread apart in the integer grid (and such that, for $u \neq v$, the convex hulls of $M(u)$ and $M(v)$ do not intersect). For the second difficulty, it is necessary to find an auxiliary unit disk graph for the crossing of two edges that communicates l-colourability information (see Figure 5).

Our NP-completeness proofs borrow upon these ideas. Before continuing, we give the following lemma.

Lemma 28 *Suppose K_1 is a $(k+1)$-clique, K_2 is a $((l-1)(k+1))$-clique, and K_3 is a j-clique, $1 \leq j \leq k+1$. Let H be the graph formed by including all possible edges between K_1 and K_2 and between K_2 and K_3. Then H is k-improperly l-colourable, and in any k-improper l-colouring of H, any vertex of K_1 and any vertex of K_3 must receive the same colour.*

Proof. Suppose we have a k-improper l-colouring of H, let u be a vertex of K_3 and assume without loss of generality that u has colour 1. The subgraph induced by $K_1 \cup K_2$ is an $(l(k+1))$-clique, so every colour appears exactly $k+1$ times in this clique and each vertex v in $K_1 \cup K_2$ has impropriety k in $K_1 \cup K_2$. Hence, the colour 1 may not appear on the vertices of K_2. Thus, the $k+1$ vertices of the clique $K_1 \cup K_2$ that are coloured 1 are those of K_1. As the vertex u of K_3 is arbitrary, this concludes the proof. □

A.1 Unit disk k-improper l-colourability, $l \geq 3$

Our approach will generalise that of Gräf *et al.* [16] and we want to show how, given any graph G, to construct a corresponding unit disk graph $\hat{G} = (\hat{V}, \hat{E})$ which is k-improperly l-colourable if and only if G is l-colourable. The key to our approach is to generalise the auxiliary graphs. We will describe k-improperly l-colourable analogues for each of the four auxiliary graphs that they employ. We use the same embedding for the given graph G, and the unit disk graph embedding needs only a slight technical modification to accommodate a larger auxiliary graph for crossings. Here is the scheme of the proof: first, we use an embedding of G in the plane that allows us to replace, in a systematic way, its edges by well chosen unit disk graphs. Our choice of unit disk graphs ensures that the existence of an l-colouring for G is equivalent to the existence of a k-improper l-colouring of $\hat{G}$. As mentioned in Section 1.1, there are two major issues in such an approach: first, as G is not necessarily planar, there may be crossing edges. Next, as the degree of G is not bounded in terms of its clique number, we have to deal with vertices of "large degree", that is vertices whose degree is more that $6\omega(G) - 6$. These two issues are solved by using two types of auxiliary unit disk graphs, the crossing gadgets and the vertex-gadgets.

Construction of the auxiliary graphs

First, we introduce the graphs that replace the edges in an embedding of G. All of these graphs are unit disk graphs and, except for the last one, use the same embeddings as in Gräf *et al.* [16]. The remaining properties are given without proof since they generally follow immediately from the construction or a straightforward application of Lemma 28. Like in the cited reference, our construction makes frequent use of cliques. In figures, these cliques will be represented by circles using the following convention:

- a small circle with a + represents a $(k+1)$-clique;
- a large circle with a $\star$ represents a $(l-2)(k+1)$-clique; and
- a large circle with a $\times$ represents a $(l-1)(k+1)$-clique.

If cliques of other size are needed, they will be represented by a large circle with the number of vertices of the clique written in it. An edge between two cliques means that all possible edges between the two cliques are present.

Definition 29 *A (k,l)-wire of order m, denoted $W_{k,l}^m$, consists of $m+1$ $(k+1)$-cliques $WV_0, \ldots, WV_m$ and m $((l-1)(k+1))$-cliques $WC_1, \ldots, WC_m$. For each $i \in \{1, \ldots, m\}$, all members of the clique WC_i are connected to the members of both WV_{i-1} and WV_i. The cliques WV_0 and WV_m are called output cliques.*

A (k,l)-wire of order 3 is shown in Figure 2.

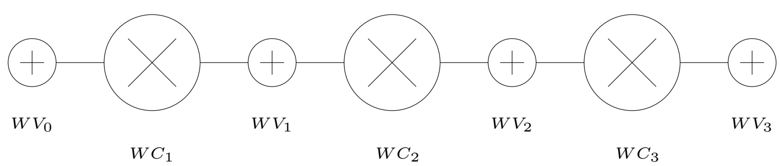


Figure 2: The (k,l)-wire $W_{k,l}^3$.

Proposition 30 *A (k,l)-wire of order m has the following properties:*

1. *$W_{k,l}^m$ has $m(l-1)(k+1) + (m+1)(k+1) = (ml+1)(k+1)$ vertices;*
2. *a (k,l)-wire is k-improperly l-colourable, but not k-improperly $(l-1)$-colourable;*
3. *each k-improper l-colouring assigns the same colour to all members of $WV_0, \ldots, WV_m$, and, in particular, the output cliques receive the same colour; and*
4. *a (k,l)-wire is a unit disk graph.*

Definition 31 *A (k,l)-chain of order m, denoted $K_{k,l}^m$, consists of a $W_{k,l}^m$ together with an additional j-clique WF connected with WV_m, for some $1 \leq j \leq (l-1)(k+1)$. The clique WV_0 is called the fixed output clique while WF is called the forced output clique.*

A (k,l)-chain of order 3 is shown in Figure 3.

Proposition 32 *A (k,l)-chain of order m has the following properties:*

1. *$K_{k,l}^m$ has $(ml+1)(k+1) + j$ vertices, where j is the size of WF;*
2. *a (k,l)-chain is k-improperly l-colourable, but not k-improperly $(l-1)$-colourable;*

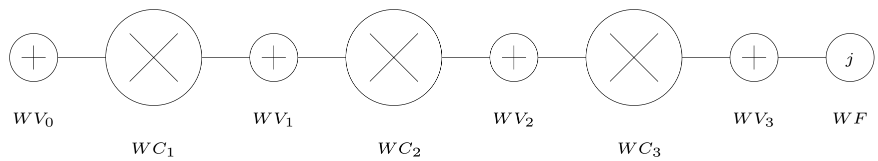


Figure 3: A (k,l)-chain of order 3 $K^3_{k,l}$.

3. *each k-improper l-colouring assigns the same colour i to all members of any of the cliques $WV_x, 1 \leq x \leq m$, and each member of the clique WF must receive a colour that is different from i;*
4. *for each pair of different colours (i_1, i_2) from the set $\{1, 2, \ldots, l\}$ there exists a k-improper l-colouring in which the forced and fixed output cliques receive colours i_1 and i_2, respectively; and*
5. *a (k,l)-chain is a unit disk graph.*

We now introduce the graphs that replace the high degree vertices of G.

Definition 33 *A (k,l)-clone of size $m \geq 2$, denoted $C^m_{k,l}$, consists of the $7m-7$ $(k+1)$-cliques $CV_1, \ldots, CV_{7m-7}$, the $7m-6$ $((l-1)(k+1))$-cliques $CC_0, \ldots, CC_{7m-7}$, and the m $(k+1)$-cliques $O_0, \ldots, O_{m-1}$. For $1 \leq i \leq 7m-7$, all members of the clique CV_i are connected to the members of both CC_{i-1} and CC_i. For each $i \in \{1, \ldots, m-1\}$, all members of O_i are connected to the members of CC_{7i}. The cliques $O_0, \ldots, O_{m-1}$ are called output cliques.*

A (k,l)-clone of size 3 is shown in Figure 4.

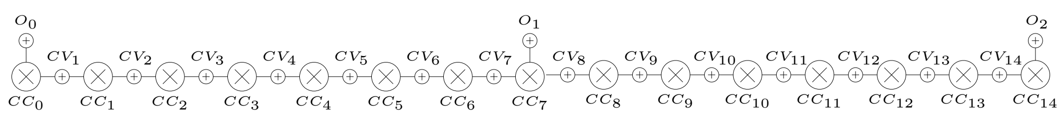


Figure 4: The (k,l)-clone $C^3_{k,l}$.

Note that, in the corresponding auxiliary graph described in Gräf *et al.* [16], every third clique was connected to an output vertex for technical reasons. For similar reasons, every seventh $((l-1)(k+1))$-clique is connected to an output clique in our construction.

Proposition 34 *A (k,l)-clone of size m has the following properties:*

1. *$C^m_{k,l}$ has $(7m-6)(l-1)(k+1) + (7m-7)(k+1) + m(k+1) = ((7m-6)l + m - 1)(k+1)$ vertices;*
2. *a (k,l)-clone is k-improperly l-colourable, but not k-improperly $(l-1)$-colourable;*
3. *each k-improper l-colouring assigns the same colour to all members of the output cliques; and*
4. *a (k,l)-clone is a unit disk graph.*

Finally, we introduce the graphs $H_{k,l}$ that replace the edge crossings in an embedding of G. This construction is based on the graph H_l used in Gräf *et al.* [16] (in the cited paper, it appears as

H_k). We replace all vertices of H_l by $(k+1)$-cliques and we replace all edges of H_l by (k,l)-chains of the appropriate order (either 1 or 2) so that the resulting graph has a unit disk representation.

When replacing edges in H_l, we have taken care to orient the (k,l)-chains so that we do not introduce cliques of size greater than $l(k+1)$; in particular, only the forced output cliques of the (k,l)-chains may be incident with the $((l-2)(k+1))$-cliques C_i of $H_{k,l}$. Note then that each $(k+1)$-clique representing a former vertex of H_l is incident to a $((l-1)(k+1))$-clique of some (k,l)-chain, and this ensures that, in a k-improper l-colouring of $H_{k,l}$, each $(k+1)$-clique is assigned a single colour.

See Figure 5 for a description of how $H_{k,l}$ is derived.

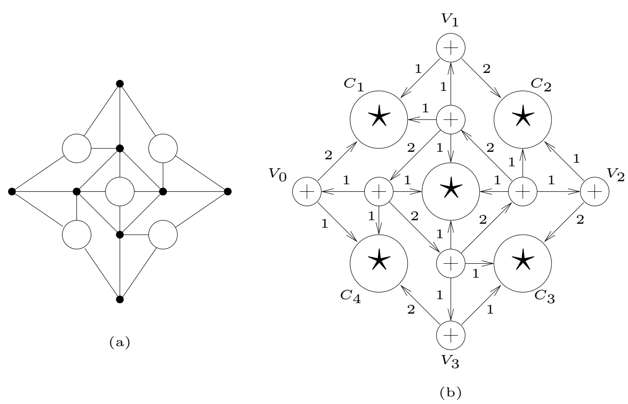


Figure 5: The derivation of the (k,l)-crossing $H_{k,l}$ from the l-crossing H_l: (a) H_l, where the circles represent $(l-2)$-cliques and (b) a schematic figure of $H_{k,l}$, where each (k,l)-chain is represented by a directed edge (the edge is directed from the fixed output vertex to the forced output vertex of the chain) together with an integer (the order of the chain).

Definition 35 *Let a (k,l)-crossing, denoted $H_{k,l}$, $l \geq 3$ be the graph in Figure 6. The cliques $V_0, \ldots, V_3$ are called output cliques.*

Proposition 36 *A (k,l)-crossing has the following properties:*

1. *$H_{l,k}$ has $(37l-2)(k+1)$ vertices;*
2. *a (k,l)-crossing is k-improperly l-colourable, not k-improperly $(l-1)$-colourable;*
3. *each k-improper l-colouring c satisfies* $c(V_0) = c(V_2)$ *and* $c(V_1) = c(V_3)$;
4. *there exist two k-improper l-colourings c_1 and c_2 which satisfy* $c_1(V_0) = c_1(V_2) = c_1(V_1) = c_1(V_3)$ *and* $c_2(V_0) = c_2(V_2) \neq c_2(V_1) = c_2(V_3)$*; and*
5. *a (k,l)-crossing is a unit disk graph.*

Embedding of the unit disk graph

As mentioned earlier, we shall use the same embedding of the given graph G, or rather, the embedding of a graph G' obtained from G. We aim at having the following properties:

- all the edges are made of vertical and horizontal line segments;
- certain minimal distances are preserved between parallel line segments, vertices and crossings; and

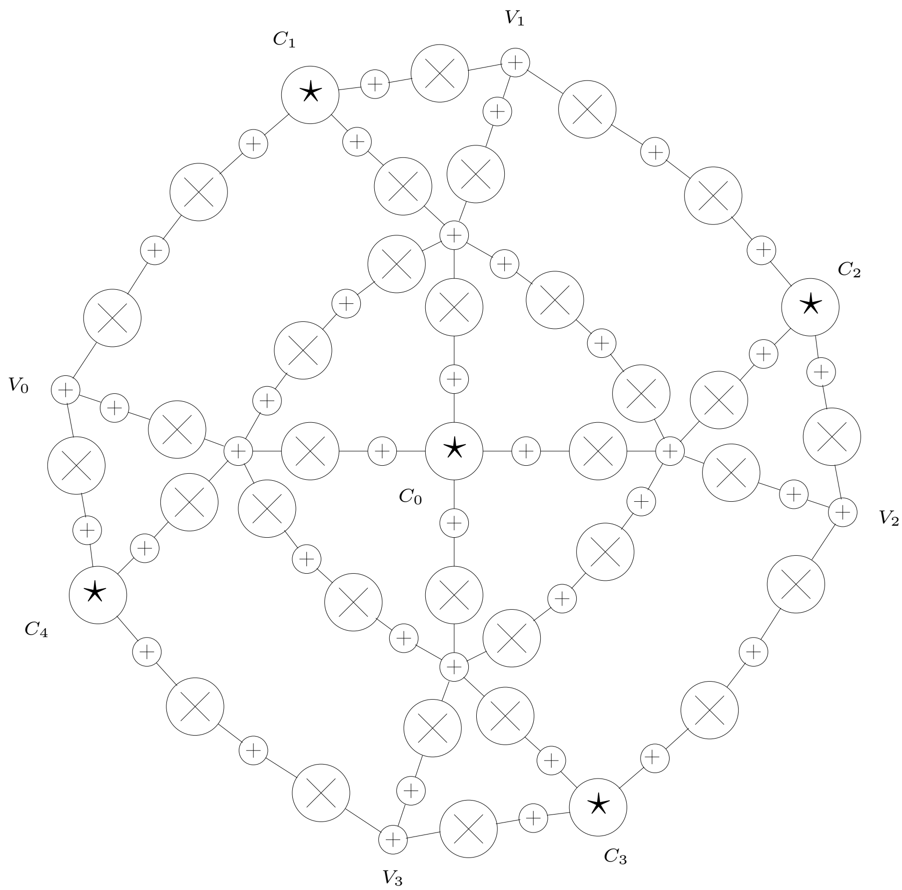


Figure 6: The (k,l)-crossing $H_{k,l}$.

- the embedding can be computed in a simple and systematic fashion.

However, for our auxiliary graphs, we must accommodate for the necessity of a larger unit disk representation for $H_{k,l}$.

First, each vertex v of G is replaced by an independent set $M(v)$ of order $d(v)$, the degree of v. Next, a vertex of $M(v)$ is linked to a vertex of $M(u)$ if and only if uv is an edge in G. The edges are added such that the maximum degree of the obtained graph $G' = (V', E')$ is one.

To describe the embedding of G', let n be an order of the vertices of V' from 1 up to $|V'|$, in a such a way that the vertices of each set $M(v)$ are numbered by consecutive integers. The vertices of G' all lie on the x-axis: the coordinates of the vertex $v \in V'$ are $(56n(v), 0)$.

An edge uv of G' is represented by the three following line segments:

$$\begin{aligned}
&\{(x, y) : x = X(u) \text{ and } y \in [0, X(u) + 8]\},\\
&\{(x, y) : x \in [X(u), X(v)] \text{ and } y = X(u) + 8\}, \text{ and}\\
&\{(x, y) : x = X(v) \text{ and } y \in [0, X(u) + 8]\}.
\end{aligned}$$

An example is given by Figure 7. Such an embedding depends on the chosen numbering n, but is unique once n is chosen.

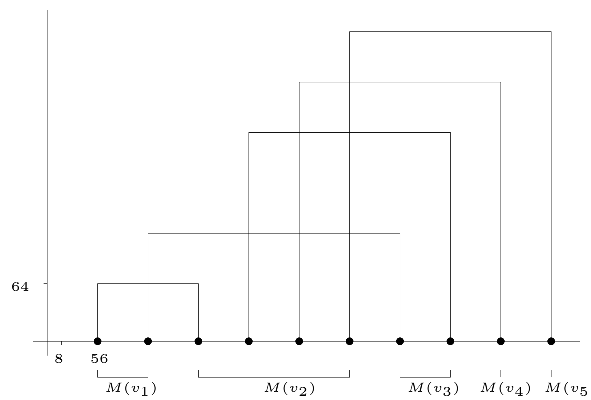


Figure 7: Embedding of the graph G'.

Let us state now the construction of $\hat{G}$, and exhibit a representation thereby showing that it is a unit disk graph.

Notice that each clique of the auxiliary graphs can be represented by a single disk, since every vertex u belongs to a clique $C(u)$, and if u and v are adjacent then all the vertices of $C(u)$ and $C(v)$ are. So, using elementary properties of disks in the plane, it is sufficient to represent each clique by a vertex, called its *representative*. We shall now give a representation of G with disks of radius 3.

The (k, l)-wires and the (k, l)-chains are embedded similarly to the l-wires and l-chains, respectively, of Gräf *et al.* [16]. These auxiliary graphs replace line segments in the embedding of G' Observe that a (k, l)-wire of order m can be embedded such that the distance between the centre of the disks of the representative of the output cliques is $8m$, and all the centres lie on a line. With a slight modification (since it contains one more clique), a (k, l)-chain of order m can also be embedded such that the distance between the representative of the output cliques is $8m$, and all the centres are on the same line. Moreover, the embedding of a (k, l)-wire can be modified such that the union of line segments joining consecutive centres makes a right angle (Figure 8).

Now follows the embedding used for a (k, l)-clone. Let (x, y) be the coordinates of the centre of the representative of the output clique O_0: the coordinates of the centres of the representative of the clique CC_i are $(x+8i, y-5)$, $i \in \{0, 1, \ldots, 7m-6\}$, those of the representative of the clique CV_i are $(x + 4i, y - 5)$, $i \in \{1, 2, \ldots, 3n - 3\}$, and those of the representative of the output clique O_i are $(x + 56i, y)$, $i \in \{1, 2, \ldots, m - 1\}$ (see Figure 9).

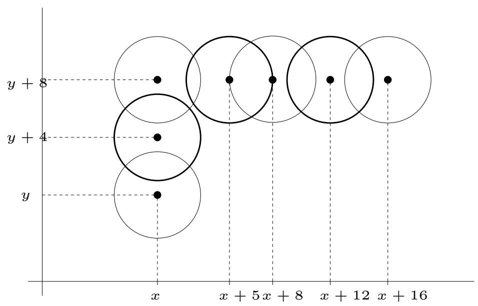


Figure 8: Embedding of a (k, l)-wire making a right angle.

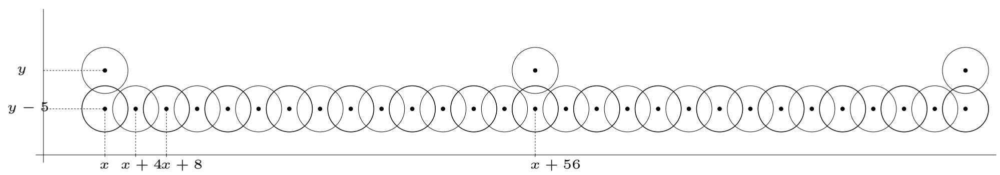


Figure 9: Embedding of a (k, l)-clone of order 3.

Figure 10 shows an embedding of a (k, l)-crossing with disks of radius 3. The centres of the representative of the output cliques lie at distance 24 of the centre.

The final steps now are not difficult: for every vertex v of G, a (k, l)-clone of order $d(v)$ is embedded in such a way that the coordinates of the centre of the representative of O_0 are $(x(v), 0)$, where $x(v)$ is the smallest x-coordinate of a vertex of $M(v)$. The coordinates of the output cliques therefore are exactly those of the vertices in $M(v)$. It only remains to replace the edges. Let uv an edge of G'. If it crosses no other edge, then a (k, l)-chain of appropriate order is embedded along the three line segments representing the edge uv, such that the representative of the output cliques identify with the vertices u and v. More precisely, if the x-coordinate of u is $8s$, and the one of v is $8s'$ with $s < s'$, then the order of the (k, l)-chain is $s + 1 + (s' - s - 2) + 1 + s = s + s'$.

If the edge uv crosses at least one other edge, then let (x, y) be the coordinates of a crossing. The four points of coordinate $(x - 24, y), (x + 24, y), (x, y - 24)$ and $(x, y + 24)$ are replaced by the output cliques of a (k, l)-crossing. The remaining line segments are all replaced by (k, l)-wires of appropriate orders, except the one containing the vertex v, which is replaced by a (k, l)-chain. This is possible since the length of each line segment is a multiple of 8.

Proof of Theorem 9

It only remains to prove that the graph G is l-colourable if and only if the graph $\hat{G}$ is k-improperly l-colourable. For each vertex v of G, let $I(v)$ be the set of vertices of the output clique of the (k, l)-clone replacing the vertex v.

Let c be an l-colouring of G. Each vertex of $I(v)$ is given the colour $c(v)$, and this colouring is extended to a k-improper l-colouring of the corresponding (k, l)-clone by Proposition 34(2) and (3). Consider now an edge uv of G. If the corresponding output cliques are linked by a single (k, l)-chain, then the colouring can be extended by Proposition 32(4), since $c(u) \neq c(v)$. Otherwise, the colouring is extended to each (k, l)-wire using Proposition 30(3), i.e. all the vertices of each output clique are assigned the same colour. Once this is done for every edge, the colouring is extended to each (k, l)-crossing by Proposition 36(4). Finally, each yet uncoloured (k, l)-chain is coloured by using Proposition 32(4), since the output cliques are coloured differently (one having the colour $c(u)$ and the other $c(v)$ for two adjacent vertices u, v of G).

Let $\hat{c}$ be a k-improper l-colouring of $\hat{G}$. Each vertex v of G is given the colour of any vertex belonging to the output clique replacing v. By the construction, and Propositions 30(3), 32(3), 34(3) and 36(3), the obtained l-colouring c of G is proper.

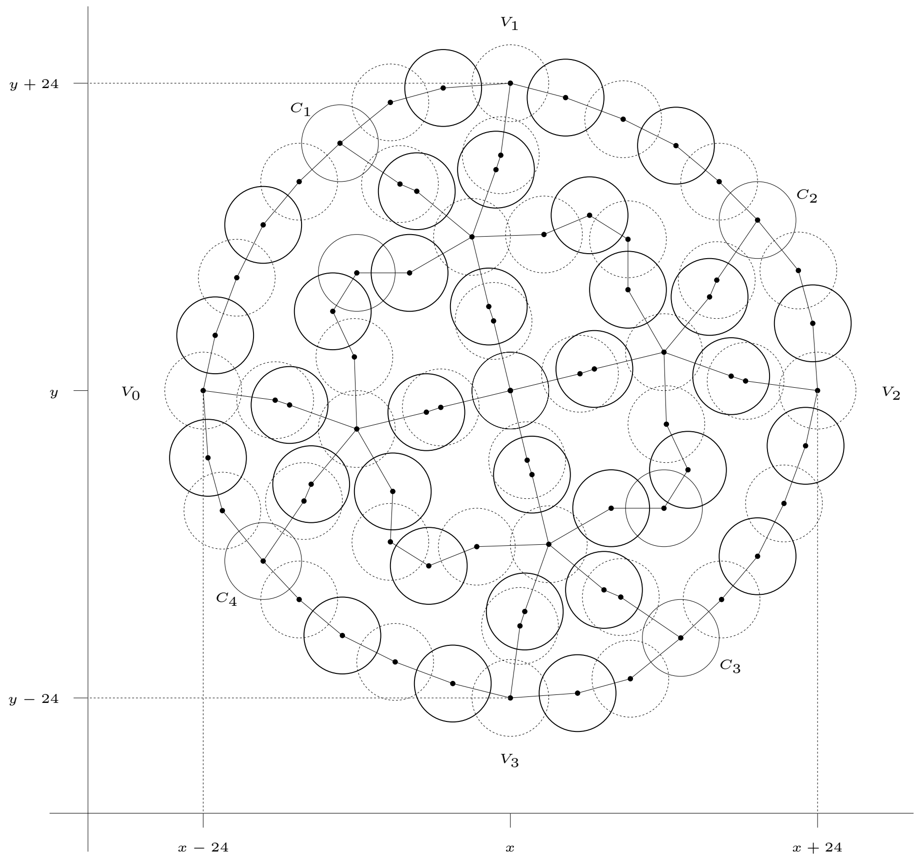


Figure 10: An embedding of the (k, l)-crossing: a bold-lined disk represents $(l-1)(k+1)$ copies of the same disk, a dash-lined disk represents $k+1$ copies of the same disk while each of the five remaining disks represents $(l-2)(k+1)$ copies of the same disk.

A.2 Unit disk k-improper 2-colourability, $k \geq 1$

For Theorem 10, our reduction is from k-improper 2-colourability of planar graphs. Given any planar graph G, we show how to construct, in polynomial time, a unit disk graph $\hat{G}$ which is k-improperly 2-colourable if and only if G is. Our construction is based on Gräf *et al.* [16], but, for the embedding, we have added the condition of planarity. Hence, we do not require a crossing auxiliary graph. On the other hand, since we are dealing entirely with k-improper 2-colouring, we must take care to handle impropriety appropriately.

Construction of the auxiliary graphs

These graphs are unit disk graphs. We give the corresponding unit disk representations later. First, we introduce the graphs that replace the edges in an embedding of G.

Definition 37 *A $(k, 2)$-bond, denoted $B_{k,2}$, has vertex set $\{v_0, \ldots, v_{2k+2}\}$. For the edge set, the vertices $\{v_1, \ldots, v_{2k+1}\}$ induce a clique, v_0 is adjacent to any $v_i, i \leq k+1$, and v_{2k+2} is adjacent to any $v_i, i \geq k+1$. The vertices v_0 and v_{2k+2} are called output vertices.*

A $(k, 2)$-bond is shown in Figure 11.

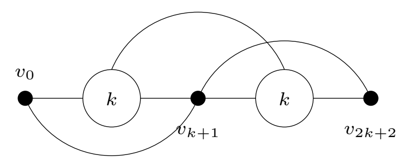


Figure 11: The $(k, 2)$-bond $B_{k,2}$.

Proposition 38 *A $(k, 2)$-bond has the following properties:*

1. *$B_{k,2}$ has $2k + 3$ vertices;*
2. *a $(k, 2)$-bond is k-improperly 2-colourable, not k-improperly 1-colourable;*
3. *each k-improper 2-colouring of $B_{k,2}$ assigns the same colour to v_0 and v_{2k+2};*
4. *suppose v_0 is adjacent to $j \in \{0, \ldots, k\}$ additional vertices $u_1, \ldots, u_j$ and furthermore suppose that $v_0, u_1, \ldots, u_j$ are precoloured with the same colour: then v_{2k+2} has impropriety at least j in any k-improper 2-colouring of $B_{k,2}$;*
5. *under the same conditions as the previous property, there exists a k-improper 2-colouring of $B_{k,2}$ such that v_{2k+2} has impropriety j; and*
6. *a $(k, 2)$-bond is a unit disk graph.*

Note that, in the case of the third and fourth properties, we say that v_0 is coloured with external impropriety j.

Proof. The first two properties immediately follow from the definition, so we focus on proving properties 3–5. Assume that c is a k-improper 2-colouring of $B_{k,2}$.

To prove the third property, suppose that $c(v_0) = 1$ and $c(v_{2k+2}) = 2$. Note that, as $\{v_1, \ldots, v_{2k+1}\}$ induces a $(2k + 1)$-clique, then one colour, say 2, must appear exactly $k + 1$ times. Hence, any such vertex coloured 2 has impropriety k in the clique, and so cannot be a neighbour of v_{2k+2}. However, among $v_1, \ldots, v_{2k+1}$, there are only k non-neighbours of v_{2k+2}. This is a contradiction.

To prove the fourth property, suppose that $c(v_0) = c(u_1) = \cdots = c(u_j) = 1$. Since v_0 has impropriety j, colour 1 appears at most $k - j$ times among $v_1, \ldots, v_{k+1}$. As $v_1, \ldots, v_{2k+1}$ is a $(2k+1)$-clique, there are at least k vertices of colour 1. We deduce that there are at least j vertices among $\{v_{k+1}, \ldots, v_{2k+1}\}$ coloured 1. Since $c(v_{2k+2}) = 1$ by the above, v_{2k+2} has impropriety at least j.

For the fifth property, again suppose that $c(v_0) = c(u_1) = \cdots = c(u_j) = 1$. Set $c(v_1) = c(v_2) = \cdots = c(v_{k-j}) = 1$. Set $c(v_{k-j+1}) = c(v_{k-j+2}) = \cdots = c(v_{2k-j+1}) = 2$. Set $c(v_{2k-j+2}) = c(v_{2k-j+3}) = \cdots = c(v_{2k+2}) = 1$. It is routine to check that this colouring satisfies our requirement.

We describe the embedding of $B_{k,2}$ in the next section. □

Definition 39 *A $(k,2)$-wire of order m, denoted $W^m_{k,2}$, is the left-to-right concatenation of m $(k,2)$-bonds $B_1, \ldots, B_m$. The extreme vertices, v_0 of B_1 and v_{2k+2} of B_m, are called output vertices.*

A $(k,2)$-wire of order 3 is shown in Figure 12. The following properties follow from Proposition 38.

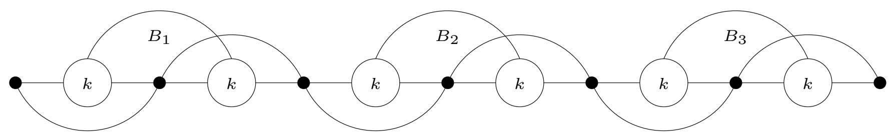


Figure 12: The $(k,2)$-wire of order 3 $W^3_{k,2}$.

Proposition 40 *A $(k,2)$-wire of order m has the following properties:*

1. *$W^m_{k,2}$ has $m(2k+2)+1$ vertices;*
2. *a $(k,2)$-wire is k-improperly 2-colourable, not k-improperly 1-colourable;*
3. *each k-improper 2-colouring of $W^m_{k,2}$ assigns the same colour to the output vertices;*
4. *if an output vertex v of B_i has external impropriety $j \in \{0, \ldots, k\}$, then, in any k-improper 2-colouring of $W^m_{k,2}$, the other output vertex of B_i has impropriety at least j;*
5. *if an output vertex v of B_i has external impropriety $j \in \{0, \ldots, k\}$, there exists a k-improper 2-colouring of $W^m_{k,2}$ such that the other output vertex of B_i has impropriety j; and*
6. *a $(k,2)$-wire is a unit disk graph.*

Definition 41 *A $(k,2)$-clone of size $m \geq 2$, denoted $C^m_{k,2}$, consists of m output vertices $o_1, \ldots, o_m$, such that there is a $(k,2)$-wire W_i between o_i and o_{i+1} for each $i \in \{1, \ldots, m-1\}$.*

A $(2,2)$-clone of size 3 is shown in Figure 13. Note that we have defined the $(k,2)$-clone to have arbitrary order, but we will apply $(k,2)$-clones of bounded order to our embedding.

Proposition 42 *A $(k,2)$-clone has the following properties:*

1. *$C^m_{k,2}$ has $l(2k+2)+1$ vertices, for some $l \geq m$;*
2. *a $(k,2)$-clone is k-improperly 2-colourable, not k-improperly 1-colourable;*
3. *each k-improper 2-colouring of $C^m_{k,2}$ assigns the same colour to all output vertices;*
4. *in any k-improper 2-colouring of $C^m_{k,2}$, the sum of external improprieties of the output vertices (cf. the remark following Proposition 38) is at most k;*

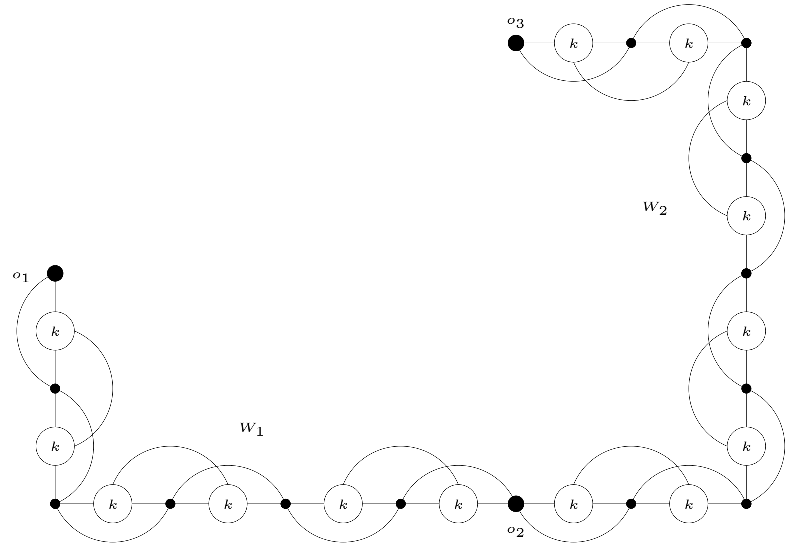


Figure 13: A $(2,2)$-clone of size 3 $C^3_{2,2}$.

5. *given a sequence $s_1, \ldots, s_m$ of non-negative integers whose sum is at most k, there is a k-improper 2 colouring of $C^m_{k,2}$ such that the external impropriety of o_i is s_i, $1 \le i \le m$; and*

6. *a $(k,2)$-clone is a unit disk graph.*

Proof. For the fifth property, we colour the vertices of $C^m_{k,2}$ starting at o_1. Suppose $c(o_1) = 1$. By Proposition 40(5), since o_1 and o_2 are output vertices of W_1, there exists a k-improper 2-colouring of W_1 such that o_2 has impropriety s_1. Now, $c(o_2) = 1$ and, if we set the external impropriety of o_2 in W_2 to $s_1 + s_2$, then o_2 has external impropriety s_2 in $C^m_{k,2}$. By Proposition 40(5), there exists a k-improper 2-colouring of W_2 such that o_3 has impropriety $s_1 + s_2$. We can carry on like this until we have coloured all of $C^m_{k,2}$, since $s_1 + s_2 + \cdots + c_m \le k$.

The other properties use similar applications of Proposition 38. $\square$

Definition 43 *For any odd positive integer m, a $(k,2)$-link of order m, denoted $K^m_{2,k}$, is defined as follows. The vertex set is $\{v_0, \ldots, v_{x(k,m)+1}\}$, where*

$$x(k,m) = \begin{cases} mk(k+1) & \text{if } k \text{ is even} \\ mk(k+1) + k + 1 & \text{if } k \text{ is odd} \end{cases}$$

For the edge set, we join v_i and v_j if and only if $|i - j| \le k + 1$. The vertices v_0 and $v_{x(k,m)+1}$ are called output vertices.

A $(2,2)$-link of order 1 is shown in Figure 14.

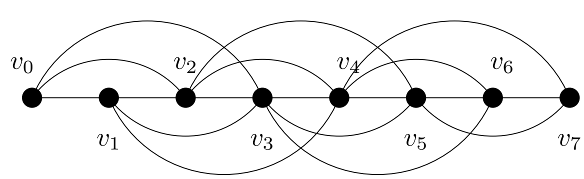


Figure 14: The $(2,2)$-link of order 1 $K^1_{2,2}$.

Proposition 44 *A $(k,2)$-link has the following properties:*

1. *$K_{k,2}$ has $x(k,m)+2$ vertices;*
2. *a $(k,2)$-link is k-improperly 2-colourable, not k-improperly 1-colourable;*
3. *for any k-improper 2-colouring of $K_{k,2}$ in which the output vertices receive the same colour, the output vertices both have non-zero improprieties;*
4. *there exists a k-improper 2-colouring of $K_{k,2}$ such that the output vertices receive different colours and both vertices have impropriety zero;*
5. *there exists a k-improper 2-colouring of $K_{k,2}$ such that the output vertices receive the same colour and both vertices have impropriety one; and*
6. *a $(k,2)$-link is a unit disk graph.*

Proof. We focus on proving properties 3–6.

For the third property, suppose c is a k-improper 2-colouring of $K_{k,2}$ such that both output vertices are coloured 1. By symmetry, suppose that v_0 has impropriety 0. Then $c(v_i)=2$ for each $i \in \{1,\ldots,k+1\}$. In particular, note that $\mathrm{im}_{\{v_1,\ldots,v_k\}}(v_{k+1})=k$, so every vertex v_i with $i \in \{k+2,\ldots,2k+2\}$ must be coloured 1. More generally, the only possibility is that $c(v_i)=1$ if and only if $(m-1)(k+1)+1 \le i \le m(k+1)$ for m an even integer. However, since $\frac{x(k,m)}{k+1}$ is even, the $k+1$ vertices with indices between $\left(\frac{x(k,m)}{k+1}-1\right)(k+1)+1$ and $x(k,m)$ are coloured $1=c(v_{x(k,m)+1})$. Since these $k+1$ vertices are adjacent to $v_{x(k,m)+1}$ we have a contradiction.

For the fourth property, we use the above forced colouring. In other words, set $c(v_0)=1$, $c(v_{x(k,m)+1})=2$ and for $1 \le i \le x(k,m)$, set $c(v_i)=1$ if and only if $(m-1)(k+1)+1 \le i \le m(k+1)$ for m an even integer. Observe that the output vertices have impropriety zero.

For the fifth property, we use the following colouring. Set $c(v_0)=c(v_{x(k,m)+1})=1$. For each $i \in \{1,\ldots,x(k,m)\}$, set $c(v_i)=1$ if and only if the index i is between $(m-1)k+1$ and mk for m an even integer. Under this colouring, v_0 is adjacent to exactly one vertex with colour 1, namely, v_{k+1}. For the impropriety of $v_{x(k,m)+1}$, we have to check the parity cases for k. If k is even, then $\frac{x(k,m)}{k}$ is odd and the only neighbour of $v_{x(k,m)+1}$ with colour 1 is $v_{x(k,m)-k}$; if k is odd, then $\frac{x(k,m)-1}{k}$ is odd and the only neighbour of $v_{x(k,m)+1}$ with colour 1 is $v_{x(k,m)}$. In either case, $v_{x(k,m)+1}$ has impropriety one.

We describe the embedding of $K_{k,2}$ in the next section. □

Definition 45 *A $(k,2)$-chain of order (m,n), denoted $K_{k,2}^{(m,n)}$, consists of the concatenation of a $(k,2)$-wire of order j ($B_1B_2\cdots B_j$) with a single $(k,2)$-link of order n (K_1) then with another $(k,2)$-wire of order $m-j$ ($B_{j+1}B_{j+2}\cdots B_m$) for some $j \in \{2,\ldots,m-1\}$. The extreme vertices, v_0 of B_1 and v_{2k+2} of B_m, are called output vertices.*

A $(2,2)$-chain of order $(2,1)$ is shown in Figure 15. The following properties follow from Propositions 38 and 44

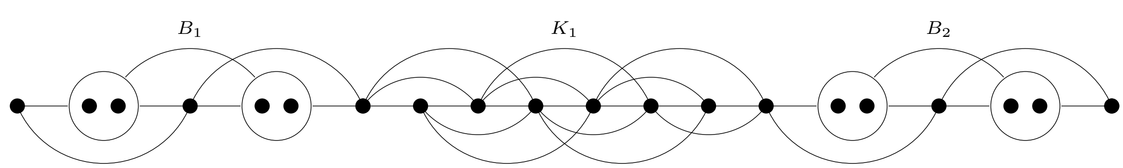


Figure 15: The $(2,2)$-chain of order $(2,1)$ $K_{2,2}^{(2,1)}$.

Proposition 46 *A $(k,2)$-chain of order (m,n) has the following properties:*

1. *$K_{k,2}^{(m,n)}$ has $m(2k+2)+x(k,m)+2$ vertices;*
2. *a $(k,2)$-chain is k-improperly 2-colourable, not k-improperly 1-colourable;*
3. *for any k-improper 2-colouring of $K_{k,2}^{(m,n)}$ in which the output vertices receive the same colour, the output vertices both have non-zero improprieties;*
4. *there exists a k-improper 2-colouring of $K_{k,2}^{(m,n)}$ such that the output vertices receive different colours and both vertices have impropriety zero;*
5. *there exists a k-improper 2-colouring of $K_{k,2}^{(m,n)}$ such that the output vertices receive the same colour and both vertices have impropriety one; and*
6. *a $(k,2)$-chain is a unit disk graph.*

Now, we introduce the graphs that replace the high-degree vertices in G.

Embedding of the unit disk graph

Given any planar graph G, we now show how to compute a distance model with distance value 1 of a unit disk graph $\hat{G}$ which is k-improperly 2-colourable if and only if G is k-improperly 2-colourable. First, we embed G in the plane in a suitable way. Then we construct $\hat{G}$ so that the vertices and edges of the original graph are replaced by the auxiliary graphs described above. Because of the definition of the $(k,2)$-chain, there are naturally two different classes of unit disk embeddings depending on the parity of k. We only fully describe the case of even k since the other case is similar.

Let $G=(V,E)$ be a planar graph. As for Theorem 23, we generate a box-orthogonal embedding of G. Let us assume that no two edges meet at a point, i.e. no box is degenerate and no two edges meet at the corner of a box. (We can do this by expanding each box by distance 1/2 in each of the four directions then doubling the scale of the grid).

Each vertex $v \in V$ is replaced by a box $Box(v)$, and we let $M(v)$ be the set of the $\deg(v)$ points of contact with edges. We aim to embed a $(k,2)$-clone in the perimeter of $Box(v)$ such that its output vertices replace the vertices in $M(v)$. We can do this by starting at an arbitrary point of $M(v)$ and proceed in clockwise direction about the perimeter. We extend the $(k,2)$-clone with an embedding of a $(k,2)$-wire to the next grid point in the perimeter and continue until all members of $M(v)$ have been included. It only remains to describe the unit disk embedding of some $(k,2)$-wire between two adjacent grid points.

Each edge $e \in E$ is replaced by a line $A(e)$ consisting of alternate horizontal and vertical line segments of the grid. It follows that $A(e)$ has integer grid length. We aim to embed a $(k,2)$-chain along $A(e)$. Since we use $(k,2)$-wires to extend a $(k,2)$-chain to arbitrary length, it suffices to describe the unit disk embedding of some $(k,2)$-chain between two adjacent grid points.

We first describe unit disk embeddings for the elementary auxiliary graphs: the $(k,2)$-bonds and $(k,2)$-links.

Let E_K^m be the embedding of a $(k,2)$-link of order m. Each centre of the disk replacing a vertex of $K_{k,2}^m$ lies on a line. The points are distributed equidistant from each other. Let the distance between adjacent vertices v_i and v_{i+1} be $d=\frac{mk}{mk(k+1)+1}$. Since $(k+2)^{-1} \le d < (k+1)^{-1}$, v_i is adjacent to v_j if and only if $|i-j| \le k+1$. Also, one can check that the distance in E_K^m between output vertices is precisely mk. See Figure 16.

We use two different embeddings for the $(k,2)$-bonds. In the first embedding E_B^a, the disks for the output vertices of $B_{k,2}$ are touching (but not intersecting) and, hence, the distance between the output vertices is 1. This embedding is illustrated in Figure 17(a). Note that the two bold disks represent cliques of size k. In the second embedding E_B^b, all of the disks lie on a line. The output vertices of $B_{k,2}$ are at distance $2-2d'$, where $d'=\frac{1}{k+3}$ and the central vertex v_{k+1} of $B_{k,2}$

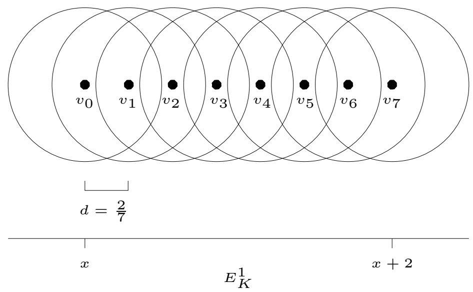


Figure 16: An embedding of the $(2, 2)$-link of order 1.

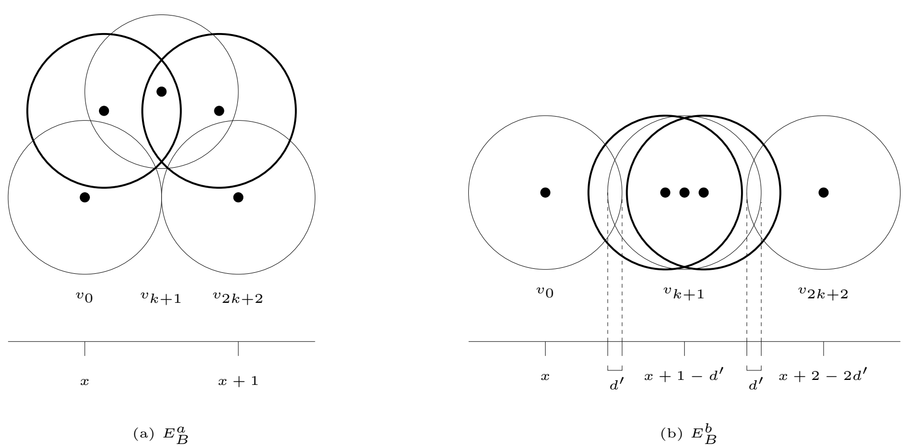
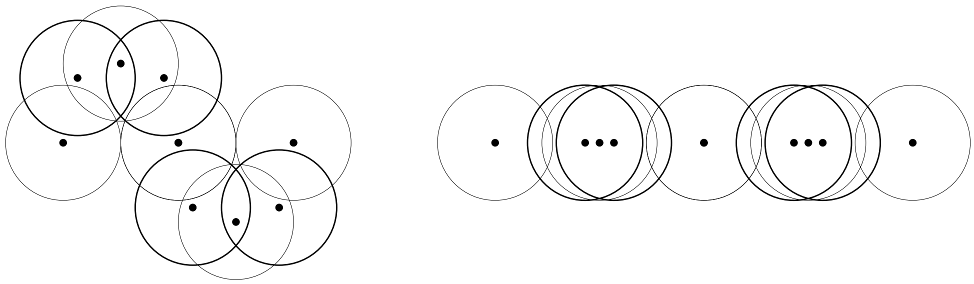


Figure 17: Two embeddings of a $(k, 2)$-bond: (a) E_B^a and (b) E_B^b.

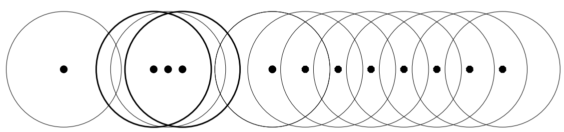
is midway between them. The centres of the two k-clique disks are at distance $1 - \frac{d+d'}{2}$ from the nearer respective output vertices. See Figure 17(b).

E_B^a can be concatenated with itself, as can E_B^b. See Figure 18. Also, E_B^b can be concatenated with E_K^m and with E_B^a. See Figure 19.

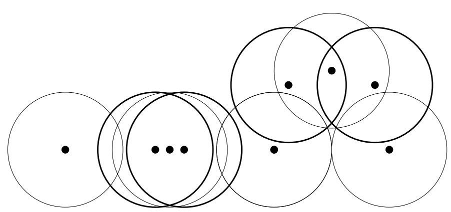


(a) $E_B^a E_B^a$ (b) $E_B^b E_B^b$

Figure 18: (a) The concatenation of two copies of E_B^a and (b) of two copies of E_B^b.



(a) $E_B^b E_K^1$



(b) $E_B^b E_B^a$

Figure 19: (a) The concatenation of E_B^b with E_K^1 and (b) of E_B^b with E_B^a.

We use these constructions to show that there are embeddings of some $(k, 2)$-wire and of some $(k, 2)$-chain between two adjacent grid points. We first scale the grid so that two adjacent grid points are distance $u = 5k+8$ apart. We embed a $(k, 2)$-wire W^* of order $3k+6$ by concatenating $k+3$ copies of E_B^b with k copies of E_B^a with $k+3$ more copies of E_B^b. This embedding has length $(k+3)(2-2d') + k + (k+3)(2-2d') = 5k+8$, as required. We embed a $(k, 2)$-chain K^* of order $2(k+3)+1$ by concatenating $k+3$ copies of E_B^b with E_K^1 with $k+3$ more copies of E_B^b. This embedding has also has length $5k+8$, as required.

Since, in E_B^b, the distance between an output vertex and any other vertex is at least $\frac{1}{2}(\frac{2}{3}+\frac{3}{4}) > \frac{1}{\sqrt{2}}$, E_B^b can be concatenated with a perpendicular copy of itself (to bend around corners). See Figure 20. Now, for each vertex v, we embed W^* between grid points along the perimeter of $Box(v)$ to obtain a $(k, 2)$-clone whose output vertices are precisely $M(v)$. Also, for each edge e we embed W^* between grid points along $A(e)$, except for one pair of grid points between which we embed a K^*, to obtain an embedding of a $(k, 2)$-chain along $A(e)$. The resulting graph is $\hat{G}$.

We remark that for the case of odd k, we choose the values $d = \frac{mk+1}{(mk+1)(k+1)+1}$, $d' = \frac{1}{k+3}$ and $u = 5k+9$.

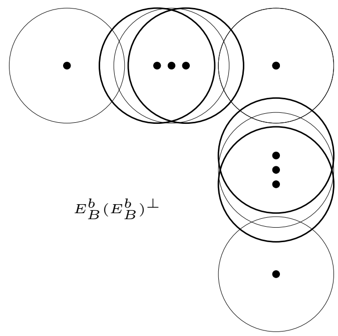


Figure 20: The embedding of $W^2_{k,2}$ around a right-angle turn.

Proof of Theorem 10

Let $G = (V, E)$ be a planar graph. One can verify that the construction of the corresponding unit disk graph $\hat{G}$ and its embedding can be performed in polynomial time. It remains to show that G is k-improperly 2-colourable if and only if $\hat{G}$ is. Each vertex $v \in V$ is replaced by a box, and the set of points of contact with edges is $M(v)$. These points are then replaced by the output vertices of a $(k, 2)$-clone if $|M(v)| \geq 2$. The set of output vertices is $I(v) = \{v\}$ if $|M(v)| = 1$.

Let c be a k-improper 2-colouring of G. We want to construct a k-improper 2-colouring of $\hat{G}$. First, for each vertex v of G, we colour the vertices of $I(v)$ by $c(v)$.

Second, for each edge $e = xy$ of G, let K_e be the $(k, 2)$-chain that connects $I(x)$ to $I(y)$ in $\hat{G}$. If $c(x) \neq c(y)$, then we apply Proposition 46(4) to colour K_e. If $c(x) = c(y)$, then we apply Proposition 46(5).

Last, for each vertex v of G, let C_v be the $(k, 2)$-clone whose output vertices are $I(v)$. Since c is a k-improper 2-colouring, we can apply Proposition 42(5) to colour C_v. In this way, we obtain a k-improper 2-colouring of $\hat{G}$.

Conversely, let $\hat{c}$ be a k-improper 2-colouring of $\hat{G}$. We want to construct a k-improper 2-colouring of G. By Proposition 42(3), for any vertex v of G, we can assign the colour of the vertices of $I(v)$. By Proposition 42(4) and Proposition 46(3), the colouring generated is a k-improper 2-colouring of G.

A.3 Improper colouring of hexagonal graphs

Proof of Theorem 13. We shall generalise the proof of McDiarmid and Reed [26] and reduce the problem from 3-colourability of planar graphs with maximum degree 4. Suppose we are given a planar graph G with maximum degree 4. We construct an induced subgraph F of the triangular lattice T and a corresponding weight vector w such that G is 3-colourable if and only if (F, w) admits a weighted k-improper 3-colouring.

The construction of F is the same as in McDiarmid and Reed [26], and we recall it here for completeness: for every $v \in T$, let H be the subgraph of T induced by all the vertices at distance at most 3 from v. The infinite face of H is bounded by a regular hexagon; the “contact points” of H are the six extreme points of this hexagon. For each vertex v of G, we make a copy H_v of the hexagon H, and place them suitably far apart in the lattice. For each edge $e = uv$ of G, we put an induced path P^e between one of the contact points of H_u and one of the contact points of H_v. It is possible to make these paths P^e completely disjoint. Furthermore, we can suppose that they all have odd length. First subdivide each edge e of G by adding a vertex ν_e. Then observe that between any two contact points of the hexagon H_ν there are induced paths of both parities within the hexagon. Thus, we are always able to make only odd-length paths.

Now we define the weight w to be $k+1$ for each vertex of F, except for every second internal vertex of each path P^e for which we give weight $2k+2$. We show that F_w is k-improperly 3-

colourable if and only if G is 3-colourable. In any k-improper 3-colouring of the hexagon H_w, each of the $(k+1)$-cliques corresponding to a vertex $v \in H$ is coloured with a single colour; moreover, all the $(k+1)$-cliques induced by the contact points have the *same* colour. It follows by applying Lemma 28 that the only k-improper 3-colouring of H_w (up to permutations of colours) is the one induced by the only proper 3-colouring of H.

In any k-improper 3-colouring of a path P_w^e, Lemma 28 shows that the $(2k+2)$-cliques induced by every second internal vertex each uses exactly two colours, and any remaining $(k+1)$-clique uses exactly one colour, which is the same for all of them. Thus, the terminal vertices must be coloured differently (and the extremal cliques must have only one colour since they both belong to a hexagon H).

So any k-improper 3-colouring of (F, w) induces a proper 3-colouring of G by assigning to any vertex v of G the colour assigned to any of the contact points of H_v in F. Conversely, given a proper 3-colouring c of G, if we assign to each contact point of H_v the multiset $\{c(v)^{k+1}\}$, it is possible to extend this partial colouring in a k-improper 3-colouring of (F, w). These observations end the proof. □

Proof of Theorem 16. We reduce to 3-colourability of planar graphs with maximum degree 4. Given a planar graph G with maximum degree 4, we construct a weighted induced subgraph (F, w) of the triangular lattice such that G is 3-colourable if and only if (F, w) is distinct k-improperly 3-colourable. For each vertex v of G, we create a copy H_v^k of an induced subgraph H^k of the triangular lattice that has four "contact points". For each edge $e = uv$ in G, we join a contact point of H_u^k to one of H_v^k by an induced subgraph P_e^k, and the two points in common with H_u^k and H_v^k are referred to as the "endpoints" of P_e^k. For a valid reduction, we ensure the following properties:

- the construction can be embedded in the lattice as an *induced* subgraph; in particular, none of the P_e^k's intersect and H_v^k can be embedded with up to four embedded P^k's leaving its contact points;
- in any distinct k-improper 3-colouring of (F, w), all the contact points of any graph H_v^k are given the same weighted colouring (i.e. set);
- in any distinct k-improper 3-colouring of (F, w), the endpoints of any graph P_e^k are given different (not necessarily disjoint) weighted colourings.

If these conditions are satisfied, we are close to done. We first need to indicate the weights of the contact points. Due to the nature of our constructions, we distinguish the case $k = 2$. All of the contact points of H^2 (hence endpoints of P^2) have weight 1. The proper colouring of $v \in G$ corresponds to the colouring applied to the contact points of $H_v^2 \in (F, w)$. For all other $k (> 0)$, the contact points of H^k have weight 2, and the proper colouring of $v \in G$ corresponds to the complement of the weighted colouring applied to the contact points of $H_v^k \in (F, w)$. So now we describe the constructions P_e^k and H^k.

Each graph P_e^k is composed of a concatenation of subgraphs E^k called "enforcers" that force their endpoints to have the same colour in any distinct k-improper 3-colouring, and of one subgraph R^k called a "reverser" that forces its endpoints to be coloured differently. Generally, the subgraph H^k is a simple extension of the enforcer E^k. For each $k \in \{1, \ldots, 5\}$, we shall now present the graphs E^k, H^k and R^k.

In our figures, a vertex is represented by a triangle if its weight is 1, a disk if its weight is 2 and a square if its weight is 3. The vertices in dash-lined circles are the contact points. We indicate the unique k-improper 3-colouring (up to permutation of the colours) of some of the enforcers, and the superscript is the impropriety of the vertex with respect to this colour. For each colour $z \in \{1, 2, 3\}$, we define $\bar{z}$ to be the colour set $\{1, 2, 3\} \setminus \{z\}$.

One can check that all the 2-weighted vertices of the enforcer E^1 must be coloured the same in any distinct 1-improper 3-colouring. Furthermore, as H_1 is just a concatenation of five enforcers, all the contact points of H_1 must be coloured the same and have impropriety 0 in H^1. So all of the

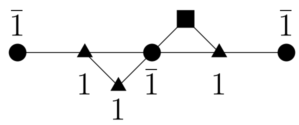


Figure 21: The enforcer E^1.

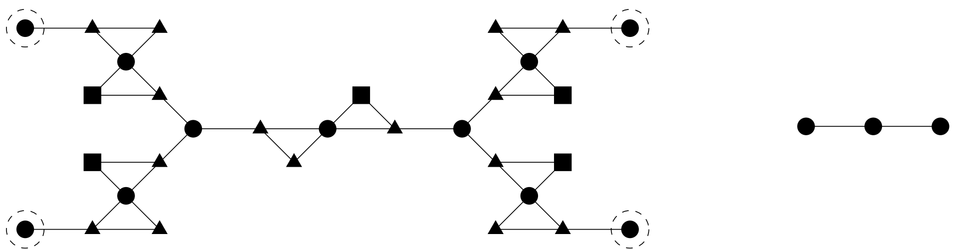


Figure 22: The induced subgraphs H^1 (on the left) and R^1.

announced properties are fulfilled, and a 12-subdivision of the triangular lattice suffices to ensure we get an induced subgraph of the triangular lattice.

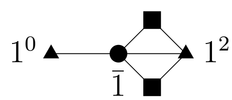


Figure 23: The enforcer E^2.

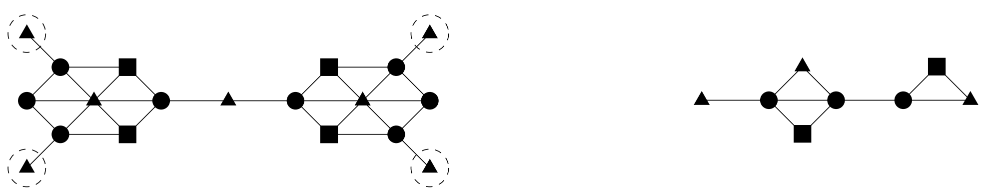


Figure 24: The induced subgraphs H^2 and R^2.

It is possible to concatenate the enforcers E^2 in straight line as well as turn 120-degree corners. The weighted graph H^2 has only one distinct 2-improper 3-colouring, in which any 1-weighted vertex is coloured 1 and any 2-weighted vertex is coloured $\bar{1}$. Moreover, each contact point has impropriety 0. As the enforcer has length 2 and the reverser has length 4, a 4-subdivision of T allows us to construct an induced subgraph of the triangular lattice.

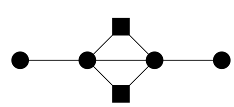


Figure 25: The enforcer E^3.

In any distinct 3-improper 3-colouring of the enforcer E^3, the endpoints have impropriety 0 with respect to one of their two colours and impropriety 1 with respect to the other. In the only 3-improper 3-colouring of H^3, all the contact points must be coloured the same and have impropriety 0 with respect to one of their two colours and impropriety 1 with respect to the other.

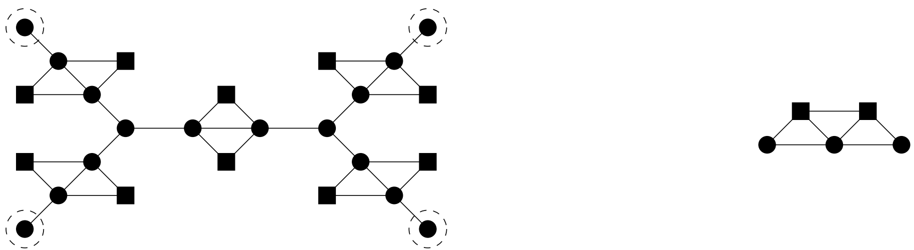


Figure 26: The induced subgraphs H^3 and R^3.

The endpoints of the reverser have impropriety 1 with respect to one of their two colours, and 2 with respect to the other. Furthermore, one can check that we can concatenate them so as to make corners. Hence, we are able to create the graphs P_e^3.

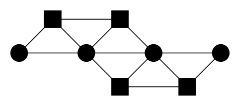


Figure 27: The enforcer E^4.

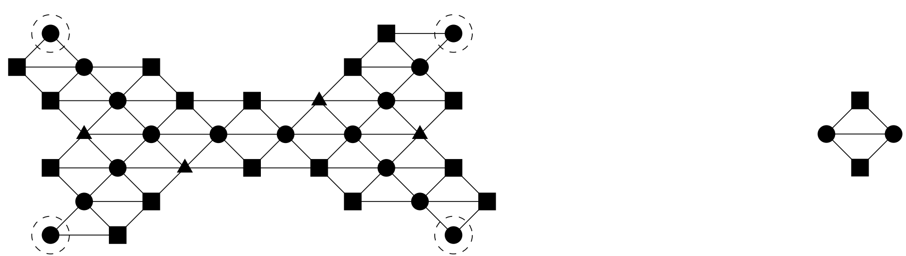
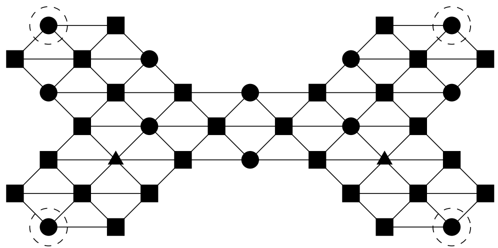


Figure 28: The induced subgraphs H^4 and R^4.

Note that in any distinct 4-improper 3-colouring of E^4, the endpoints are coloured the same, and have impropriety 1 with respect to one of their two colours and 2 with respect to the other one. The graph H^4 is an extension of the enforcer E^4. In any distinct 4-improper 3-colouring, the contact points must be coloured the same. Moreover, assuming the contact points are coloured $\bar{3}$, you can choose each to have impropriety 1 with respect to the colour 1 and impropriety 2 with respect to the colour 2, or the opposite. When adding a reverser at the contact point of a graph H^4, the other endpoint of the reverser cannot be coloured the same as the contact point since otherwise the contact has impropriety 5 with respect to one of its two colours (and 4 with respect to the other).

Figure 29: Turning with enforcers when $k = 5$.

In any distinct 5-improper 3-colouring of E^5, the endpoints are coloured the same and both have impropriety 3 in E^5 with respect to each of their two colours. Note that we can make corners when concatenating enforcers as shown in Figure 29. In any distinct 5-improper 3-colouring of H^5, the contact points are coloured the same and have impropriety 3 in H^5 with respect to each of



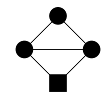


Figure 30: The induced subgraphs H^5 and R^5.

their two colours. Observe that, to concatenate H^5 and E^5, they must share a 3-weighted vertex. Each time we use the reverser, its leftmost endpoint has impropriety 3 with respect to each of its colours in the induced subgraph obtained by removing the vertices of the reverser. So one can check that the reverser actually acts like a reverser, i.e. its endpoints cannot be coloured the same. Furthermore, both have impropriety 1 with respect to one of their two colours and impropriety 2 with respect to the other. □

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